Chapter 3.6: Chain Rule

Rules For Derivates

$$\frac{d}{dx}(1) = 0$$

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(c \cdot f(x)) = c \cdot f'(x)$$

$$\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$$

$$\frac{d}{dx}(f(x) \cdot g(x)) = f(x)'g(x) + \cdot f(x)g'(x)$$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

Chain rule:

$$\boxed{\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)}$$

First Example and Idea

$$\boxed{\frac{d}{dx}\left(f(g(x))\right) = f'(g(x)) \cdot g'(x)}$$

- Differentiate the outside
- Plug the inside
- Multiply by derivative on inside

Example:

$$\frac{d}{dx}\left[\sin(x^3)\right] =$$
$$= \cos(x^3) \cdot 3x^2$$

$$\frac{d}{dx}\left[\cos(x^2) - e^{x^2}\right] =$$
$$= -\sin(x^2) \cdot 2x - e^{x^2} \cdot 2x$$

Proof idea:

$$\frac{d}{dx}\left(f(g(x))\right) = \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h}$$
$$= \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \frac{g(x+h) - g(x)}{h}$$
$$= f'(g(x)) \cdot g'(x)$$

Examples $(f(g(x)))' = f'(g(x)) \cdot g'(x)$

$$\frac{d}{dx}\left(e^{31x}\right) = e^{31x} \cdot 31 = 31e^{31}$$

$$\frac{d}{dx}\left((e^x)^{31}\right) = 31 \cdot (e^x)^{30} \cdot e^x = 31e^{31}$$

$$\frac{d}{dx}\left(\sin(\cos(\sqrt{x}))\right) =$$

$$\cos(\cos(\sqrt{x})) \cdot \frac{d}{dx}\left(\cos(\sqrt{x})\right) = \cos(\cos(\sqrt{x})) \cdot (-\sin(\sqrt{x})) \cdot \frac{1}{2}x^{-1/2}$$

$$\frac{d}{dx}(|x|) = \frac{d}{dx}\left(\sqrt{x^2}\right) = \frac{1}{2}\left(x^2\right)^{-\frac{1}{2}} \cdot \left[x^2\right]' = \frac{1}{2}\left(x^2\right)^{-\frac{1}{2}} 2x = \frac{x}{|x|}$$

$$\frac{d}{dx}\left(e^{x\sqrt{x^6+1}}\right) =$$

$$e^{x\sqrt{x^6+1}} \cdot \left[x\sqrt{x^6+1}\right]' = e^{x\sqrt{x^6+1}} \cdot \left[1 \cdot \sqrt{x^6+1} + x\left(\sqrt{x^6+1}\right)'\right] =$$

$$e^{x\sqrt{x^6+1}} \cdot \left[\sqrt{x^6+1} + x\left(\tfrac{1}{2}\left(x^6+1\right)^{-\frac{1}{2}}\left(6x^5+0\right)\right)\right]$$

Reciprocal and Quotient Rules Using Chain Rule

Reciprocal rule $\quad (f(g(x)))' = f'(g(x)) \cdot g'(x)$

$$\boxed{\frac{d}{dx}\left[\frac{1}{f(x)}\right] = \frac{-f'(x)}{f(x)^2}}$$

$$\frac{d}{dx}\left[\frac{1}{f(x)}\right] = \frac{d}{dx}\left[f(x)^{-1}\right] = -1 \cdot f(x)^{-2} \cdot f'(x) = \frac{-f'(x)}{f(x)^2}$$

Quotient rule

$$\boxed{\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}}$$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{d}{dx}\left[f(x) \cdot g(x)^{-1}\right] = f'(x)g(x)^{-1} + f(x) \cdot (-1)g(x)^{-2}g'(x)$$

$$= g(x)^{-2}\left(f'(x)g(x) - f(x)g'(x)\right) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

Magic Table $\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$

Given the following table

x=	1	2	3	4
f(x)=	3	4	2	1
f'(x)=	4	3	1	2
g(x)=	2	4	3	1
g'(x)=	2	1	4	3

Determine $h'(x)$ and $k'(x)$ for $x = 1, 2, 3, 4$ given that $h(x) = f(g(x))$ and $k(x) = g(f(x))$.

We need to compute the derivatives first:

$$h'(x) = (f(g(x))' = f'(g(x)) \cdot g'(x)$$

$$k'(x) = (g(f(x))' = g'(f(x)) \cdot f'(x)$$

x=	1	2	3	4
f'(g(x))=	3	2	1	4
g'(x)=	2	1	4	3
f'(g(x)) g'(x)=	6	2	4	12
g'(f(x))=	4	3	1	2
f'(x)=	4	3	1	2
g'(f(x))f'(x)=	16	9	1	4

More Examples $\frac{d}{dx}\left(f(g(x))\right) = f'(g(x)) \cdot g'(x)$

$$\frac{d}{dx}\left(e^{12x^2+\cos(\frac{1}{x})}\right) = e^{12x^2+\cos(\frac{1}{x})} \cdot \left(24x - \sin\left(\frac{1}{x}\right) \cdot (-1x^{-2})\right)$$

$$\frac{d}{dx}\left(\cos(\sin(x^2+e^x)) \cdot \tan(2x^3+\pi)\right) =$$

$$-\sin(\sin(x^2+e^x))\cos(x^2+e^x)(2x+e^x)\tan(2x^3+\pi) + \cos(\sin(x^2+e^x))\frac{1}{\cos(2x^3 +}$$

Find a tangent line to $y = e^{x^4-2x+1}$ at $x = 1$

$y' = e^{x^4-2x+1}(4x^3-2)$

$y'(1) = e^0(4-2) = 2$

Slope is 2, point $[1, 1]$

$y = mx + b$ hence $1 = 2 + b$ gives $b = -1$.

Hence $y = 2x - 1$ is the desired tangent line.

Chapter 3.6 Recap

$$\boxed{\frac{d}{dx}\left(f(g(x))\right) = f'(g(x)) \cdot g'(x)}$$